Optimum Techniques for the Conversion of Space Rectangular and Curvilinear Coordinates

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***Abstract*—Conversion between space rectangular (X, Y, Z) and curvilinear (φ, λ, h) coordinates is an important task in the field of Surveying, geodesy, positioning, navigation, mapping etc. Different techniques which include iterative methods, non-iterative techniques and closed form algebraic methods have been applied over the years to carry out the coordinate conversion. However, the results obtained using these techniques are deficient in one way or the other due to the inherent limitations such as inability to produce results for curvilinear coordinates when the values of X, Y and Z are subsequently or simultaneously equal to zero. Therefore, this study attempts to put forth an optimum coordinate conversion technique between space rectangular and curvilinear coordinates. The data used are coordinates of points which include the space rectangular coordinates and their equivalent curvilinear coordinates. They were observed and processed in Nigeria using Doppler 9 software by African Doppler Survey (ADOS) and they were confirmed to be of first order accuracy and hence of high quality. The data processing involved the design of the optimum techniques equations, coding of the algorithms and necessary computations to obtain results. Analyzing the results obtained, it can be inferred that the designed optimum model has successfully carried out the conversion between space rectangular and curvilinear coordinates. Therefore, the optimum technique model is recommended for use for the conversions from Space rectangular coordinates to Geocentric, Geodetic, Reduced coordinates and vice versa.**

***Index Terms*—Coordinates Conversion, Curvilinear Coordinates, Optimum Techniques, Space Rectangular Coordinates.**

I. Introduction

Coordinate conversion can be defined as the process of establishing relationship between coordinate systems in order to transform coordinates of points from one system to the other.

Coordinates and coordinate systems are ubiquitous in virtually all aspects of surveying and mapping. Importantly, they allow the users of spatial data to easily conceptualize coordinates with respect to some convenient coordinate system. As such, both coordinates and coordinate systems have had and will continue to have an essential role to play in the spatial sciences. This is because coordinate system forms a common frame of reference for the description of locations of points. Accordingly, coordinates and positions can be used interchangeably but always refer to a specific coordinate system [1]. In geodesy, two common classes of coordinate system have been used to describe positions in relation to the earth. These comprise the curvilinear coordinate system and the Cartesian coordinate system. Historically, curvilinear coordinates are used since they are conceptually more appropriate for describing positions on or near the earth's curved surface. However, space rectangular coordinates have taken an increasing role because of the widespread use of Global Positioning Systems (GPS) and other satellite-based positioning systems [1]. The review of related literature has shown that there are several techniques such as the iterative, non-iterative and closed form methods that had been developed for the conversions of the coordinates but it has been discovered that majority of these techniques suffers one limitation or the other. While some of the techniques are good for the computations of longitude and latitude, they are not satisfactory for the computation of ellipsoidal heights. More so, it was discovered that some of these techniques can only give result for conversion to curvilinear geodetic coordinates only if the values of X-coordinate and Y-coordinate are not simultaneously equals to zero. Likewise, the existing techniques cannot compute for curvilinear coordinates when the values of space rectangular coordinates (X, Y and Z) are all zeros. With the aforementioned limitations, it has become imperative to develop a new technique that will reduce the observed limitations in the earlier techniques to the barest minimum. Therefore, it is the objectives of this study to develop an optimum coordinate conversion technique for space rectangular and curvilinear coordinates.

II. Methodology

Algorithms for the Optimum techniques of conversion between space rectangular and curvilinear coordinates were designed and used to convert space rectangular coordinates to curvilinear coordinates and vice versa.

A. Data Acquisition

The data used consists of coordinates of ten points which were observed and processed in Nigeria using software Doppler 9 by African Doppler Survey (ADOS) [5] as shown in Table 1.

TABLE I: Acquired Data for the Study

STN	X(m)	Y(m)	Z(m)
	φ (deg mins secs)	λ (deg mins secs)	h (m)
ANI 001	6 151 936.18	1 333 437.29	1 026 087.83
	N 09 19 09.938	E 12 13 47.023	281.58
ANI 002	6 293 421.52	715 424.95	748 771.60
	N 06 47 13.104	E 06 29 07.589	210.96
ANI 003	6 192 908.86	740 204.22	1 333 628.71
	N 12 08 54.544	E 06 48 57.289	771.32
ANI 004	6 285 784.60	921 321.54	565 631.10
	N 05 07 19.168	E 08 20 18.943	100.54

Published on October 29, 2019.
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ANI 005	6 331 914.32 N 06 22 31.491	299 334.87 E 02 42 23.707	703 535.43 23.88
ANI 006	6 080 492.56 N 11 51 11.506	1 416 787.03 E 13 06 58.309	1 301 587.09 351.80
ANI 008	6 247 222.03 N 10 45 38.514	498 243.54 E 04 33 35.782	1 183 077.65 345.04
ANI 009	6 203 094.00 N 12 41 33.126	505 175.19 E 04 39 21.057	1 392 320.44 349.96
ANI 010	6 324 696.64 N 04 16 19.170	674 496.14 E 06 05 14.246	471 945.48 18.24
ANI 011	6 211 406.20 N 09 47 07.207	974 306.14 E 08 54 52.697	1 077 100.03 1 416.20

B. Data Quality

The quality of data used in any experiment is a function of the precision and accuracy of such data. Precision is a measure of validity while accuracy determines the reliability of the data. The coordinates of these points were observed and processed in Nigeria using software Doppler 9 by ADOS and they were confirmed to be highly precise and accurate and hence of high quality while the reference ellipsoid used for their acquisition has the following parameters: semi-major axis (a) = 6378145.0m, and flattening (f) = 1/298.25 [5].

C. Data Processing

This involves the following:

1) Conversion from Space Rectangular Coordinate to Curvilinear Coordinate

This involves three stages as follows:

a) Computation of Longitude

$$\lambda = arctan(\frac{Y}{X}) \quad (1)$$

for when X, Y and Z are greater than zero [3]

$$\lambda = \frac{\pi}{2} - 2\arctan\left[\frac{X}{\sqrt{(X^2+Y^2)}} + Y\right] \quad (2)$$

When X or Y is equal to zero [8]

$$\lambda = arctan\left(\frac{1}{\sqrt{X^2+Y^2}} - XY\right) \quad (3)$$

When X and Y are zero [9]

$$\lambda = -2Arctan\left(\frac{XY}{\sqrt{X^2+Y^2}} - XY\right) \quad (4)$$

When X, Y and Z are zero [9]

b) Computation of Latitude

In this technique, equations are solved iteratively as shown below [2]

$$g(T) = \mathrm{PT} - \mathrm{Z} - \frac{ET}{\sqrt{1+T^2}} \quad (5)$$

The initial value for T is approximated as T_o

$$T_o = \frac{|z|}{e_c^2 P} \quad (6)$$

Where T is computed as tanβ, $P = \frac{p}{a}$, $Z = \frac{e_c|z|}{a}$, E = e^2 ,

$$e_c = \sqrt{1 - e^2}$$

$$p = \sqrt{X^2 + Y^2} \quad (7)$$

The resulting iterative formula is processed as:

$$T_n + 1 = T_n - \frac{g(T_n)}{g'(Tn) - g"(Tn)g(Tn)/(2g\prime(Tn))} \quad (8)$$

The first derivative for g (T) is computed as:

$$g'(T) = P - \frac{E}{\sqrt{(1+T^2)}^3} \quad (9)$$

The second derivative for g (T) is computed as:

$$g"(T) = \frac{3ET}{\sqrt{(1+T^2)}^5} \quad (10)$$

Hence, the geodetic latitude (φ) is computed as:

$$\varphi = \sin(z)\ arctan\left(\frac{T}{e_c}\right)[2] \quad (11)$$

When X, Y, Z are greater than zero and X or Y is equal to zero

$$\varphi = arctan\frac{p}{(^b/_a)^2 r} \quad [9] \quad (12)$$

For when only X and Y are equal to zero and when X, Y and Z are equal to zero

c) Computation of Ellipsoidal Heights

$h = \frac{e_c p_G + |z_G|T - b\sqrt{1+T^2}}{\sqrt{e_c^2+T^2}}$, When X, Y and Z are greater than zero and when X or Y is equal to zero. The approximate value for height h = 0 [2] (13)

$h = |\Delta r|\sqrt{1 + (\frac{\Delta z}{\Delta r})^2}$, if $|\Delta z| \le |\Delta r|$ when X and Y are simultaneously zero and when X, Y and Z are all equal to zero [6] (14)

and

$$h = |\Delta z|\sqrt{(\frac{\Delta z}{\Delta r})^2 + 1} \quad ,\ \text{if } |\Delta r| > |\Delta z| \quad (15)$$

Where

$$\Delta r = R - r \quad (16)$$

r = N$cos\varphi$ (17)

$$R = (N + h)cos\varphi \quad (18)$$

$$\Delta z = Z - z \quad [6] \quad (19)$$

$$z = (\frac{b}{a})^2 N sin\varphi \quad (20)$$

2) Conversion from Curvilinear Coordinate to Space Rectangular Coordinate

The procedure for the conversion of curvilinear coordinate to space rectangular coordinate are shown as follows [4]

$$X = (N + h)\cos\varphi\cos\lambda \quad (21)$$

$$Y = (N + h)\cos\varphi\sin\lambda \quad (22)$$

$$Z = [N(1 - e^2) + h]\sin\varphi \quad (23)$$

Where $N = \frac{a}{(1-e^2(\sin\phi)^2)^{1/2}}$ (24)

$$e^2 = \frac{a^2-b^2}{a^2} = 2f - f^2 \quad (25)$$

3) Conversion within Geocentric (ψ), Geodetic (ϕ) and Reduced Latitude (β)

$\tan\psi = (1 - e^2)\tan\phi = {}^{b}/_{a}\tan\beta$ [7] (26)

III. Presentation of Results

The results obtained include results of conversions from space rectangular coordinates to curvilinear coordinates when the values of X, Y and Z are greater than and subsequently or simultaneously equals to zero. Also, it includes the results of reverse conversion from curvilinear coordinates to space rectangular coordinates.

TABLE II: Conversion from Space Rectangular Coordinates to Curvilinear Coordinates

STN	Input (X, Y, Z)	Converted (φ, λ, h)	Standard (φ, λ, h)	Differences
ANI 001	X=6151936.18 Y=1333437.29 Z=1026087.83	φ=09 19 9.93746 λ=12 13 47.0234 h=281.5763	φ=09 19 9.938 λ=12 13 47.023 h=281.57 8	0.00000015 0.00000011 0.003730
ANI 002	X=6293421.52 Y=715424.95 Z=748771.60	φ=06 47 13.1038 λ=06 29 7.58849 h=210.95806	φ=06 47 13.104 λ=06 29 7.589 h=210.96	0.00000006 0.00000014 0.001940
ANI 003	X=6192908.86 Y=740204.22 Z=1333628.71	φ=12 08 54.5442 λ=06 48 57.2884 h=771.32233	φ=12 08 54.544 λ=06 48 57.289 h=771.32	0.00000006 0.00000017 0.002330
ANI 004	X=6285784.60 Y=921321.54 Z=565631.10	φ=05 07 19.1683 λ=08 20 18.943 h=100.54311	φ=05 07 19.168 λ=08 20 18.943 h=100.54	0.00000008 0.00000000 0.003110
ANI 005	X=6331914.32 Y=299334.87 Z=703535.43	φ=06 22 31.491 λ=02 42 23.7067 h=23.882354	φ=06 22 31.491 λ=02 42 23.707 h=23.88	0.00000000 0.00000008 0.002354
ANI 006	X=6080492.56 Y=1416787.03 Z=1301587.09	φ=11 51 11.5061 λ=13 06 58.3087 h=351.80099	φ=11 51 11.506 λ=13 06 58.309 h=351.80	0.00000003 0.00000008 0.000990
ANI 008	X=6247222.03 Y=498243.54 Z=1183077.65	φ=10 45 38.5139 λ=04 33 35.7822 h=345.04508	φ=10 45 38.514 λ=04 33 35.782 h=345.04	0.00000003 0.00000000 0.005080
ANI 009	X=6203094.00 Y=505175.19 Z=1392320.44	φ=12 41 33.1259 λ=04 39 21.057 h=349.95171	φ=12 41 33.126 λ=04 39 21.057 h=349.96	0.00000003 0.00000000 0.001710
ANI 010	X=6324696.64 Y=674496.14 Z=471945.48	φ=04 16 19.1702 λ=06 05 14.246 h=18.239575	φ=04 16 19.170 λ=06 05 14.246 h=18.24	0.00000006 0.00000000 0.000425
ANI 011	X=6211406.20 Y=974306.14 Z=1077100.03	φ=09 47 7.20689 λ=08 54 52.6971 h=1416.2035	φ=09 47 7.207 λ=08 54 52.697 h=1416.2 0	0.00000003 0.00000003 0.003500

TABLE III: Conversion from Curvilinear Coordinates to Space Rectangular Coordinates

STN	Input (φ, λ, h)	Converted (X,Y,Z)	Standard (X,Y,Z)	Differences
ANI 001	φ=09 19 9.938 λ=12 13 47.023 h=281.58	X=6151936.18410 Y=1333437.27820 Z=1026087.84710	X=6151936.18 Y=1333437.29 Z=1026087.83	0.00410 0.0118 0.0171
ANI 002	φ=06 47 13.104 λ=06 29 07.589 h=210.96	X=6293421.51970 Y=715424.96583 Z=748771.60769	X=6293421.52 Y=715424.95 Z=748771.60	0.00030 0.01583 0.00769
ANI 003	φ=12 08 54.544 λ=06 48 57.289 h=771.32	X=6192908.85760 Y=740204.23652 Z=1333628.70320	X=6192908.86 Y=740204.22 Z=1333628.71	0.00240 0.01652 -0.0068
ANI 004	φ=05 07 19.168 λ=08 20 18.943 h=100.54	X=6285784.59840 Y=921321.53831 Z=565631.09068	X=6285784.60 Y=921321.54 Z=565631.10	0.00160 0.00169 -0.00932
ANI 005	φ=06 22 31.491 λ=02 42 23.707 h=23.88	X=6331914.31760 Y=299334.87877 Z=703535.43068	X=6331914.32 Y=299334.87 Z=703535.43	0.00240 0.00877 0.00068
ANI 006	φ=11 51 11.506 λ=13 06 58.309 h=351.80	X=6080492.55790 Y=1416787.03880 Z=1301587.08790	X=6080492.56 Y=1416787.03 Z=1301587.09	0.00210 0.0088 -0.0021
ANI 008	φ=10 45 38.514 λ=04 33 35.782 h=345.04	X=6247222.02530 Y=498243.53357 Z=1183077.65280	X=6247222.03 Y=498243.54 Z=1183077.65	0.00470 0.00643 0.0028
ANI 009	φ=12 41 33.126 λ=04 39 21.057 h=349.96	X=6203094.0080 Y=505175.19203 Z=1392320.44400	X=6203094.00 Y=505175.19 Z=1392320.44	0.00180 0.00124 0.004
ANI 010	φ=04 16 19.170 λ=06 05 14.246 h=18.24	X=6324696.64140 Y=674496.13983 Z=471945.47413	X=6324696.64 Y=674496.14 Z=471945.48	0.00140 0.00017 -0.00587
ANI 011	φ=09 47 07.207 λ=08 54 52.697 h=1416.20	X=6211406.19690 Y=974306.13667 Z=1077100.03270	X=6211406.20 Y=974306.14 Z=1077100.03	0.00310 0.00333 0.0027

TABLE IV: Converted Curvilinear Coordinates When the Values of X, Y and Z are Subsequently and Simultaneously Zero

STN	Input (X,Y,Z)	Converted *(φ, λ, h)*
ACS 01	X=0	φ=38 17 11.6442

	Y=1333437.29 Z=1026087.83	λ=90 00 00 h=-4687541.0232
ACS 02	X=6151936.18 Y=0 Z=1026087.83	φ=09 19 9.9375 λ=00 00 00 h=-140642.9366
ACS 03	X=6151936.18 Y=1333437.29 Z=0	φ=00 00 00 λ=12 13 47.0234 h=-161814.9661
ACS 04	X=0 Y=0 Z=1026087.83	φ=90 00 00 λ=90 00 00 h=6460154.0133
ACS 05	X=0 Y=0 Z=0	φ=00 00 00 λ=00 00 00 h=00 00 00

IV. Analysis of Results

The results obtained after the conversion from curvilinear coordinates to space rectangular coordinates and vice versa are analyzed as shown in the following steps:

The aim here is to test whether the converted coordinates (μ) satisfactorily represent the standard coordinates(μ_o) of the same points. That is, the following hypothesis is tested:

The null hypothesis is $H_o : \mu = \mu_o$
Alternative hypothesis is $H_1 : \mu > \mu_o$
At significance level $\alpha = 0.05$
The computed statistic is given as $t = \frac{\mu - \mu_o}{S/\sqrt{n}}$

Where t is referred to as student-t distribution statistic, μ is the mean of converted coordinates, μ_o is the mean of the standard coordinates, S is the standard deviation (SD) of the converted coordinates and n is the total number of data.

Reject H_o if $t > t_{1-\alpha}$
Accept H_o if $t < t_{1-\alpha}$

Draw appropriate conclusion based on the acceptance or rejection of Null Hypothesis.

A. Statistical evaluation for the converted Space Rectangular Coordinates

1) Test for Easting Coordinates (X)

Mean of standard Eastings, μ_o = 6232287.691
Mean of converted Eastings, μ = 6232287.68971
Standard Deviation, SD= 75910.6747647
$t = 0.00000005374$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$
$t_{0.95, 9}$ = 1.833 (from the statistical table)

Since $t = 0.00000005374 < 1.833$, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted easting coordinates.

2) Test for Northing Coordinates (Y)

Mean of Standard Northings, μ_o, = 807873.091
Mean of converted Northings μ = 807873.093774
Standard Deviation, SD = 341260.48043241
$t = 0.0000000257$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$
$t_{0.95, 9}$ = 1.833 (from table)

Since $t = 0.0000000257 < 1.833$, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted northing coordinates.

3) Test for Height Coordinates (Z)

Mean of Standard Heights, μ_o = 980368.536
Mean of Computed Heights, μ = 980368.537088
Standard Deviation, SD = 317960.93565255
$t = 0.00000001082$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$
$t_{0.95, 9}$ = 1.833 (from table)

Since $t = 0.00000001082 < 1.833$, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted height coordinates.

B. Statistical Evaluation of converted Curvilinear Coordinates

1) Test for Latitude Coordinates (φ)

Mean of Standard Latitude, μ_o = 8.911605
Mean of converted Latitude, μ = 8.911606
Standard Deviation, SD= 2.90874812
$t = 0.000001087$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$
$t_{0.95, 9}$ = 1.833 (from table)
Since $t = 0.000001087 < 1.833$, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted latitude coordinates.

2) Test for Longitude Coordinates (λ)

Mean of Standard Longitude, μ_o = 7.39101783
Mean of converted Longitude, μ =7.39101780
Standard Deviation, SD= 3.15754177
$t = -0.00000003004$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$
$t_{0.95, 9}$ = 1.833 (from table)
Since $t = -0.00000003004 < 1.833$, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted longitude coordinates.

3) Test for ellipsoidal height Coordinates (h)

Mean of Standard Ellipsoidal heights, μ_o= 386.951
Mean of Computed Ellipsoidal heights, μ = 386.952298
Standard Deviation, SD= 400.41683151
$t = 0.000010251$
Degree of freedom = 10-1 = 9
Accept H_o if $t < t_{0.95, 9}$

DOI: http://dx.doi.org/10.24018/ejers.2019.4.10.1588

$t_{0.95,\,9}$ = 1.833 (from table)

Since t = 0.000010251 < 1.833, the null hypothesis is accepted.

Therefore, it can be concluded that there is no significant difference between the standard and the converted ellipsoidal height coordinates.

V. Conclusion

This paper has attempted to develop a new coordinate conversion techniques known as the 'Optimum Coordinate Conversion Model' for the transformation of space rectangular coordinates to curvilinear coordinates and vice versa. Based on the results obtained and the statistical analysis carried out at 95% confidence level, it can be concluded that the optimum coordinate conversion techniques developed in this study has successfully and accurately produced optimum results for the conversion of space rectangular coordinate to curvilinear coordinates and vice versa. Also, this has removed the inherent limitations observed in other techniques of coordinate conversion. Therefore, it is recommended that the optimum technique should be adopted for the conversions between space rectangular and curvilinear coordinates which are very vital in geodetic positioning, mapping and navigations all other aspects of Geodesy.

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