Four-Poles Parameter of an Elliptical Cavity Having the Outlet on the Body

Yuya Nishimura, and Sohei Nishimura

***Abstract*—This paper deals with the computation of the four-poles parameter of a thin elliptic cylinder in which the output is fitted to the side that is perpendicular to the input side. The four-poles parameter is based on the sound pressure calculated by solving the wave equations, with the assumption that the loss can be ignored. The four-poles parameter is widely used to estimate the noise characteristic for the acoustic system which are composed of several elements of various cross-sectional areas, various shape connected in series.**

***Index Terms*—Four-Poles Parameter, Higher-Order Mode, Wave Equation, Mathieu Function.**

I. INTRODUCTION

Analysis based on the wave equation is the most effective method for determining the characteristics of an acoustic system [1]-[12]. It's been nearly a century, noise control has grown into a fully-fledged science using very advanced techniques based on the wave equation and producing a thousand scientific papers each year. From the general solution of the wave equation, the sound pressure propagating inside the analysis model can be easily obtained by applying the boundary conditions. As a result, the input and output sound pressure could be obtained related to these volume velocities.

However, the drawback of this method is not suitable for complex structures. Instead, the transfer matrix comprised of the four-poles parameters characterizing the relationship between the upstream and downstream pressure and velocities is widely used in the analysis and design of silencing systems [13]-[16].

This paper deals with the computation of the four-poles parameter of a thin elliptic cylinder in which the output is fitted to the side that is perpendicular to the input side [17] as shown in Fig. 1. Following the development of the theory base on reference [17] in section 2, the calculation of the four-poles parameter is presented in section 3.

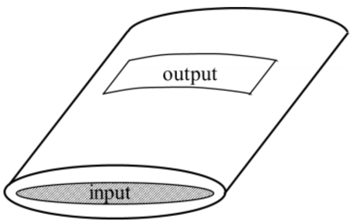

Fig. 1. The elliptic cavity

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II. THE SOUND PROPAGATION IN THE ELLIPTICAL CAVITY

A. The solution of the wave equation in elliptical coordinates

Three-dimensional wave equation in elliptical coordinate (ξ, η, z) is given as

$$\frac{2}{q^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2\Phi}{\partial\xi^2}+\frac{\partial^2\Phi}{\partial\eta^2}\right)+\frac{\partial^2\Phi}{\partial z^2}+k^2\Phi=0 \quad (1)$$

where q is the distance between the foci and the origin, Φ be the complex effective value of velocity potential, c is the sound velocity, $k=\omega/c$ (ω: angular velocity).

Then the general solution of Equation (1) becomes

$$\Phi=\left(A_0\exp(\mu z)+B_0\exp(-\mu z)\right)\left(\sum_{m=0}^{\infty}C_m Ce_m(\xi,s)ce_m(\eta,s)+\sum_{m=0}^{\infty}S_m Se_m(\xi,s)se_m(\eta,s)\right) \quad (2)$$

where

$$s=\frac{q^2}{4}(k^2+\mu^2) \quad (3)$$

A_0, B_0, C_m, S_m and μ are arbitrary constants. Function $ce_m(\eta,s)$, $se_m(\eta,s)$ and $Ce_m(\xi,s)$, $Se_m(\xi,s)$ are the Mathieu functions and the modified Mathieu function of mth-order [18], respectively.

B. Model of analysis and boundary conditions

The model of analysis is shown in Fig. 2. The elliptical cavity of eccentricity e_w and length L which has an elliptical piston-driven of eccentricity e_i fitted at the center of the input. The output which has an opening area $u\times h$ is located on the cavity and its center is at the ℓ position from the input.

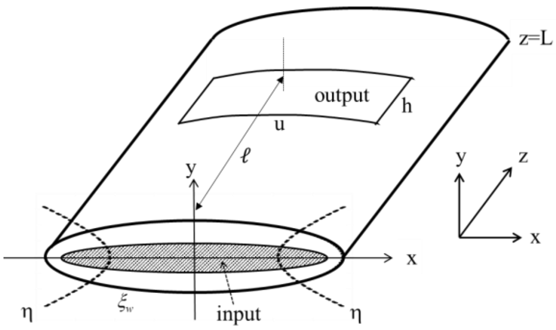

Fig. 2. Model analysis of an elliptic cavity

Since the distribution of Φ is symmetric about the major and the minor axis, Φ must be even and periodic in η, hence Equation (2) becomes

$$\Phi = (A_0 \exp(\mu z) + B_0 \exp(-\mu z)) \sum_{m=0}^{\infty} C_{2m} Ce_{2m}(\xi, s) ce_{2m}(\eta, s) \quad (4)$$

The even Mathieu and even modified Mathieu function are defined as [17]

$$Ce_{2m}(\xi, s) = ce_{2m}(j\xi, s) = \sum_{r=0}^{\infty} A_{2r}^{(2m)} \cosh 2r\xi \quad (5)$$

where $A_{2r}^{(2m)}$ are constants.

Let V_z and V_ξ are the velocity components in the z and ξ directions, V_i and V_0 are the driving velocity at the input and the output piston, respectively. The boundary conditions are as follows:

$$\text{(1) at the input } V_z = -\left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = V_i F_i(\xi, \eta) \quad (6)$$

$$\text{(2) at the output } V_\xi = -\left.\frac{\partial \Phi}{\partial \xi}\right|_{\xi=\xi_w} = V_0 F_0(\eta, z) \quad (7)$$

$$\text{(3) at the back surface } V_z = -\left.\frac{\partial \Phi}{\partial z}\right|_{z=L} = 0 \quad (8)$$

where $F_i(\xi, \eta) = 1$ at the piston and $F_i(\xi, \eta) = 0$ elsewhere. Similarly, $F_0(\eta, z) = 1$ at the piston and $F_0(\eta, z) = 0$ elsewhere.

C. *Average sound pressure of inlet and outlet*

According to the boundary conditions Equations (6)- (8), Φ can be determined as

$$\Phi = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left(\frac{H_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + G_{2m,i} U_o \right) \times \cosh \mu_{2m,i}(L-z)\, Ce_{2m}(\xi, s_{2m,i})\, ce_{2m}(\eta, s_{2m,i}) \quad (9)$$

The sound pressure propagates in the cavity corresponding to Φ becomes

$$P(\xi, \eta, z) = jk\rho c \Phi = jk\rho c \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left(\frac{H_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + G_{2m,i} U_o \right) \times \cosh \mu_{2m,i}(L-z)\, Ce_{2m}(\xi, s_{2m,i})\, ce_{2m}(\eta, s_{2m,i}) \quad (10)$$

where ρ is density, $U_i = V_i S_i$ and $U_0 = V_0 S_0$ are volume velocity at input and output, the other symbols are constants defined as following

$$\mu_{2m,i} = \frac{1}{a_w} \sqrt{\lambda_{2m,i}{}^2 - (ka_w)^2} \quad (11)$$

$$\lambda_{2m,i} = \frac{2\sqrt{s_{2m,i}}}{e_w} \quad (12)$$

$$H_{2m,i} = \frac{1}{S_i} \frac{\int_0^{\xi_i} \int_0^{2\pi} Ce_{2m}(\xi, s_{2m,i})\, ce_{2m}(\eta, s_{2m,i})(\cosh 2\xi - \cos 2\eta)\, d\eta\, d\xi}{\int_0^{\xi_w} \int_0^{2\pi} Ce_{2m}^2(\xi, s_{2m,i})\, ce_{2m}^2(\eta, s_{2m,i})(\cosh 2\xi - \cos 2\eta)\, d\eta\, d\xi} \quad (13)$$

$$G_{2m,i} = \frac{1}{S_0} \frac{2A_0^{(2m)}}{Ce_{2m}(\xi, s_{2m,i})} \frac{\int_{l-h/2}^{l+h/2} \cosh \mu_{2m,i}(L-z)\, dz}{\int_0^L \cosh^2 \mu_{2m,i}(L-z)\, dz} \quad (14)$$

The average sound pressure acting on the input piston can be expressed as

$$\overline{P}_i = jk\rho c \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left(\frac{D_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + E_{2m,i} U_o \right) \times \cosh \mu_{2m,i} L \quad (15)$$

Where

$$D_{2m,i} = H_{2m,i} \frac{q^2}{2S_i} \int_0^{\xi_i} \int_0^{2\pi} Ce_{2m}(\xi, s_{2m,i})\, ce_{2m}(\eta, s_{2m,i})(\cosh 2\xi - \cos 2\eta)\, d\eta\, d\xi \quad (16)$$

$$E_{2m,i} = H_{2m,i} \frac{q^2}{2S_i} \int_0^{\xi_i} \int_0^{2\pi} Ce_{2m}(\xi, s_{2m,i})\, ce_{2m}(\eta, s_{2m,i})(\cosh 2\xi - \cos 2\eta)\, d\eta\, d\xi \quad (17)$$

The average sound pressure acting on the output piston can be expressed

$$\overline{P}_o = jk\rho c \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \left(\frac{Q_{2m,i}}{\mu_{2m,i} \sinh \mu_{2m,i} L} U_i + R_{2m,i} U_o \right) \left(\frac{\cosh \mu_{2m,i}(\ell - L) \sinh(\mu_{2m,i} h/2)}{\mu_{2m,i}} \right) \quad (18)$$

where

$$Q_{2m,i} = H_{2m,i}\, Ce_{2m}(\xi_w, s_{2m,i})\, \chi_{2m,i} \quad (19)$$

$$R_{2m,i} = G_{2m,i}\, Ce_{2m}(\xi_w, s_{2m,i})\, \chi_{2m,i} \quad (20)$$

$$\chi_{2m,i} = \frac{a_w}{S_o} \int_{\eta_1}^{\eta_2} \sqrt{1 - e_w^2 \cos^2 \eta}\; ce_{2m}(\eta, s_{2m,i})\, d\eta \quad (21)$$

D. *The computation of four-poles parameter*

The four-poles parameter A, B, C, and D are defined as follows [18]

$$A = \left.\frac{\overline{P}_i}{\overline{P}_0}\right|_{U_0=0} \quad (22)$$

$$B = \left.\frac{\overline{P}_i}{U_0}\right|_{\overline{P}_0=0} \quad (23)$$

$$C = \left.\frac{U_i}{\overline{P}_0}\right|_{U_0=0} \quad (24)$$

$$D = \frac{U_i}{U_0}\bigg|_{\bar{P}_0 = 0} \qquad (25)$$

The parameter A and C can be obtained by setting $U_0 = 0$ in Equation (15) and Equation (18), respectively.

$$A = \frac{\bar{P}_i}{\bar{P}_0}\bigg|_{U_0=0} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{D_{2m,i}}{Q_{2m,i}} \frac{\mu_{2m,i}\cosh\mu_{2m,i} L}{\cosh\mu_{2m,i}(\ell - L)\ \sinh(\mu_{2m,i}\, h/2)} \qquad (26)$$

$$C = \frac{U_i}{\bar{P}_0}\bigg|_{U_0=0} = \frac{1}{jk\rho c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{Q_{2m,i}} \frac{\mu^2_{2m,i}\sinh\mu_{2m,i} L}{\cosh\mu_{2m,i}(\ell - L)\ \sinh(\mu_{2m,i}\, h/2)} \qquad (27)$$

when $\bar{P}_0$ of Equation (22) becomes zero, we have

$$\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\left(\frac{Q_{2m,i}}{\mu_{2m,i}\sinh\mu_{2m,i} L} U_i + R_{2m,i}\, U_o \right) = 0$$

Thus, the parameter D is

$$D = \frac{U_i}{U_0}\bigg|_{\bar{P}_0=0} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{R_{2m,i}}{Q_{2m,i}} \mu_{2m,i} \sinh\mu_{2m,i} L \qquad (28)$$

Similarly, from Equation (21) we can obtain parameter B as follows

$$B = \frac{\bar{P}_i}{U_0}\bigg|_{\bar{P}_0=0} = jk\rho c \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{D_{2m,i} R_{2m,i}}{Q_{2m,i}} + E_{2m,i} \right) \cosh\mu_{2m,i} L \qquad (29)$$

III. Application

Consider the case of the ventilation system having two elements connected as shown in Fig.3. The element-1 is an elliptical cavity as mentioned above and the element-2 is a rectangular cubic that has a dimension of $a_2 \times b_2 \times L_2$, respectively. Let A, B, C, and D are the four-poles parameter of element-1 and A_2, B_2, C_2, and D_2 are the four-pole parameters of element-2 then the four-pole parameters of the ventilation system becomes

$$\begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\bar{H}^{0,0}_{m,n}}{\bar{H}^{0,L}_{m,n}} \cosh\tilde{\mu}_{m,n} L_2 & jk\rho c \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\bar{H}^{0,L}_{m,n} \sinh\tilde{\mu}_{m,n} L_2}{\tilde{\mu}_{m,n}} \\ \frac{1}{jk\rho c}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\tilde{\mu}_{m,n}\sinh\tilde{\mu}_{m,n} L_2}{\bar{H}^{0,L}_{m,n}} & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\bar{H}^{L,L}_{m,n}}{\bar{H}^{0,L}_{m,n}} \cosh\tilde{\mu}_{m,n} L_2 \end{bmatrix} \qquad (30)$$

where

$$\bar{H}^{r,s}_{m,n} = \frac{1}{S_r S_s} \frac{\iint_{S_r} \cos\left(\frac{m\pi x}{a_2}\right)\cos\left(\frac{n\pi y}{b_2}\right) dxdy \iint_{S_s} \cos\left(\frac{m\pi x}{a_2}\right)\cos\left(\frac{n\pi y}{b_2}\right) dxdy}{\iint_{S_2} \cos^2\left(\frac{m\pi x}{a_2}\right)\cos^2\left(\frac{n\pi y}{b_2}\right) dxdy} \qquad (31)$$

$$\tilde{\mu}_{m,n} = \sqrt{(m\pi / a_2)^2 + (n\pi / b_2)^2 - k^2} \qquad (32)$$

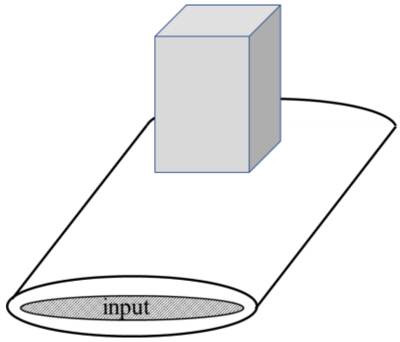

Fig. 3. Model of system in series

IV. Conclusion

A derivation of the four-poles parameter for the elliptical cavity having an output on the body has been presented by solving the wave equation on the assumption that the loss at the wall can be neglected. The four-poles parameter is given by Equation (26) to Equation (29) including the higher-order mode wave. The formula derived in the present study enables us to account for the characteristic of acoustic system which is composed of several elements of different form, different cross-sectional areas in series. It is also utilized to determine the acoustic characteristic of the most acoustic system, such as the muffler, to any high-frequency range.

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