Contribution to the Development of a Two-Dimensional Asymptotic Theory of the Three-Point Bending Behaviour of Multi-Layered Beams: Applications to Orthotropic Phase Sandwich Beams

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***Abstract* — The objective of this work is to present a methodology for analyzing the behavior in bending of the structure of sandwich beams base on the second order of asymptotic method. This work is in continuation with the work of Talla [1]. This work includes the knowledge of all the physical elastic constant of the sandwich beams. This result confirms the fact that the second order of asymptotic method doesn't bring a significative change in the behavior of the solution until a certain point. The curves have been obtained by the software named python. This result was predictable because the asymptotic methods deal with small variation due to the presence of the epsilon parameter, which is very small.**

***Index Terms* — Asymptotic method, Composite material, Python, Sandwich beams.**

I. Introduction

In practice, however, an approximation of a solution to such problems is usually enough. Thus, the approaches to finding such an approximation is important. There are two main methods. One is numerical approximation, which is especially powerful after the invention of the computer and is now regarded as the third most import method for scientific research (just after the traditional two: theoretical and experimental methods). Another is analytical approximation with an error which is understandable and controllable, in particular, the error could be made smaller by some rational procedure.[2].

The module *Asymptotic Methods in Mechanics* consists of analytical approximation techniques for solving algebraic, ordinary and partial differential equations and corresponding initial and boundary value problems with a small parameter arising in mechanics and other areas of sciences, engineering and technology [3].

Essentially new methods of asymptotic integration of differential equations with a small parameter in the higher derivative term has also been developed by M. I. Vyshyk and L. A. Lyusternik [4].

Sandwich structures have received considerable attention recently, primarily because of their high specific stiffness and strength properties. These structures are typically composed of two thin composite laminated faces and a thick soft core made of foam or low-strength honeycomb. Sandwich construction has been used in aircraft, marine, and other types of structures [5].

The objectives of our investigations are the evaluation the behaviour of a orthotropic material using the second other of the asymptotic theory. In order to verify if the result is different from the first order and compare both. In this study, we adopted an experimental approach to evaluate the physical properties of this species exploited in southeastern Cameroon and the software named python to bring out the curves.

II. Material And Methods

A. Material

1. Presentation of Specie Under Study Species Descriptions

Ayous or Obéché, the commercial name for our wood specie. This species belongs to the Sterculiaceae family, with the scientific name Triplochiton Scleroxylon K. Schum. Statistics from the ATIBT (International Technical Association of Tropical Timber) between July 2001 and June 2002 reveal that this species ranks 1st just before Tali, Sapelli and Iroko, in terms of export volume (440618 m^3) at the port of Douala (Cameroon). As main uses, it is in great demand as wood for interior joinery, moulding and plywood. Geographically, Ayous is mainly found on the African continent, particularly in West and Central Africa [6].

Iroko or Abang, the commercial name for our wood species. This species belongs to the Moraceae family and has the scientific name Chlorophora (Milicia) excelsa. Statistics from the ATIBT (International Technical Association of Tropical Timber) between July 2001 and June 2002 reveals that this species ranks 4th just after Sapelli in terms of export volume (73215 m^3) at the port of Douala (Cameroon). As main uses, it is in great demand as wood for exterior and

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interior joinery, furniture, parquet flooring, stairs, decorative veneer, shipbuilding. Geographically, Iroko is mainly found on the African continent, especially in West Africa, Central Africa, East Africa [6].

B. Method

1. Variationnel Formulation of Elasticity Problem in Three Dimensions

Let us consider for this purpose a volume (Ω) of material of boundary Γ sufficiently regular. The border is divided into Γ_σ where we have applied the forces of surfaces T_1 and Γ_u, where we imposed displacement $\bar{u}$. With $\Gamma_\sigma \cup \Gamma_u = \Gamma$ and $\Gamma_\sigma \cap \Gamma_u = \Phi$.

Where we associated the following boundary conditions:

$$\begin{cases} \sigma\vec{n} = \vec{T} & on \quad \Gamma_\sigma \\ u = \bar{u} & on \quad \Gamma_u \end{cases} \tag{1}$$

In order to apply the integral method, let us first state some spaces, which we will use in this work: Sobolev spaces, Lebesgue spaces, Hilbert spacesю

The Law of behavior and the Equilibrium equation gives us the following equations:

$$a(\sigma,T) = B(T,u) \tag{2}$$

$$B(\sigma,V) = F(V) \tag{3}$$

Let us notice that the equations above include all the boundary conditions.

2. *Introduction to the Small Dimensionless Parameter ε Definition of the Field*

Let us consider the plane field Ω^ε defined by: $\Omega^\varepsilon =]-\varepsilon,\varepsilon[\times]0,L[$ with the boundary $\Gamma^\varepsilon = \Gamma_\sigma^\varepsilon \cup \Gamma_u^\varepsilon$ with $\Gamma_\sigma^\varepsilon = \Gamma_\sigma^{+\varepsilon} \cup \Gamma_\sigma^{-\varepsilon}$ and $\Gamma_u^\varepsilon = \Gamma_u^{+\varepsilon} \cup \Gamma_u^{-\varepsilon}$ [7].

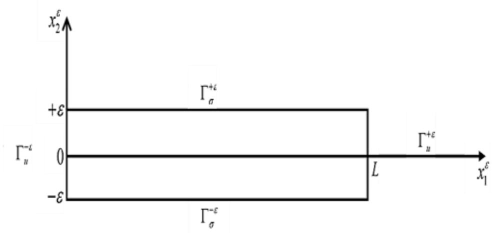

Fig. 1. Loaded beam bending.

3. *Asymptotic Development of the Solution*

The aim of the solving is to find an approximate solution to a physical (mechanical) problem depending on a small dimensionless parameter epsilon. This solution is obtained in the form of a development called (asymptotic) as a function of this small epsilon.

4. *Formal Development of (σ, u)*

This equation below leads us to believe that the solution of the couple (σ, u) of the problem admits a development of the form.

$$\begin{cases} a_0(\sigma,T) + \varepsilon^2 a_2(\sigma,T) + \varepsilon^4 a_4(\sigma,T) + B(T,U) = 0 \\ B(U,V) = F(V) \end{cases} \tag{4}$$

$$(\sigma,u) = (\sigma^0,u^0) + \varepsilon^2(\sigma^2,u^2) + \varepsilon^4(\sigma^4,u^4) + ... \tag{5}$$

Nothing justifies this development yet. However, we will see that this development is suitable thanks to the conditions to which it leads us.

Carrying from equation (5) in equations (4) the problem of elasticity gives us the following equations by simplification and identification of terms having the same power of ε on both side of equality, we obtain:

$$(A)\begin{cases} a_0(\sigma^0,T) + B(T,u^0) = 0 \\ B(\sigma^0,v) = F(v) \end{cases} \tag{6}$$

$$(B)\begin{cases} a_0(\sigma^2,T) + a_2(\sigma^0,T) + B(T,u^2) = 0 \\ B(\sigma^2,v) = 0 \end{cases} \tag{7}$$

$$(C)\begin{cases} a_0(\sigma^4,T) + a_2(\sigma^2,T) + a_4(\sigma^0,T) + B(T,u^4) = 0 \\ B(\sigma^4,v) = 0 \end{cases} \tag{8}$$

The terms (σ^0, u^0) translate the global behavior of the solution while the others translate the local behavior of (σ, u). We will not limit ourselves here to the calculation of (σ^0, u^0) which gives a good account of the phenomenon studied, micro-rotations not being considered. Note that (σ^0, u^0) only represents an approximate solution to the problem.

We have to notice that the equation (6) has already been solved by Talla [1]. In this work we will endeavor to find the solution of (σ^2, u^2), which represented the second order of the first one. To evaluate it, we will focus our solving on the equation (B) using the solutions of equation (A) already obtained by Talla [1].

First, let's recall the expression of the deflection found by Talla.

$$u_2^0 = \sum_1^m \frac{2}{\varepsilon} \frac{L^3 T_2^\varepsilon}{\mathrm{Re}\, m^4 \pi^4} \sin\tfrac{m\pi}{L}\xi \sin\tfrac{m\pi}{L} x_1 ; \tag{9}$$

$$u_2^0 = \sum_1^m \frac{2\times 12 L^3 T_2^\varepsilon}{\varepsilon^4 m^4 \pi^4 \left(\frac{7E_1^1}{1-\nu_{12}^1\nu_{21}^1} + \frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2} \right)} \sin\tfrac{m\pi}{L}\xi \sin\tfrac{m\pi}{L} x_1 . \tag{10}$$

The deflection in the middle of the beam for a loading applied in the middle of the beam it suffices to do: $\xi = L/2$ and $x_1 = L/2$.

Therefore:

$$u_2^0 = \sum_1^m \frac{24\ L^3\ T_2^\varepsilon}{\varepsilon^4 m^4 \pi^4 \left(\frac{7E_1^1}{1-\nu_{12}^1\nu_{21}^1} + \frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2} \right)} \tag{11}$$

The slope of the below equation is given by:

$$a = \sum_1^m \frac{\varepsilon^4 m^4 \pi^4}{24 L^3} \left(\frac{7E_1^1}{1-\nu_{12}^1\nu_{21}^1} + \frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2} \right) \tag{12}$$

Now we can state the solving of our equation (B).

5. *Calculation of* (σ^2, u^2)

For the solving of (σ^2, u^2), lets us consider the equation (7), with the following values:

$$a_0(\sigma,T) = \varepsilon \int_\Omega \left(\frac{1}{E_1} \sigma_{11} T_{11} \right) d\Omega \tag{13}$$

$$a_2(\sigma,T) = \int_\Omega \left(\frac{1}{G_{12}} \sigma_{12} T_{12} - \frac{\nu_{21}}{E_2} \sigma_{22} T_{11} - \frac{\nu_{12}}{E_1} \sigma_{11} T_{22} \right) d\Omega \tag{14}$$

$$B(\sigma,V) = \int_\Omega \sigma_{ij} \frac{\partial V_i}{\partial x_j} d\Omega = \int_\Omega \sigma_{ij} T_{ij} d\Omega = \tag{15}$$

$$B(T^\varepsilon, V^\varepsilon) = \int_{\Omega^\varepsilon} T_{ij}^\varepsilon \frac{\partial V_i^\varepsilon}{\partial x_j^\varepsilon} d\Omega^\varepsilon \tag{16}$$

Equation (7) imply:

$$\int_\Omega \frac{1}{E_1} \sigma_{11}^2 T_{11} d\Omega + \int_\Omega T_{11} \frac{\partial u_1^2}{\partial x_1} d\Omega = \int_\Omega \frac{\nu_{12}}{E_1} \sigma_{22}^0 T_{11} d\Omega \ \forall T_{11} \in \underset{\sim}{\Sigma} \tag{17}$$

$$\int_\Omega T_{11} \left(\frac{\partial u_1^2}{\partial x_2} + \frac{\partial u_2^2}{\partial x_1} \right) d\Omega = -\int_\Omega \frac{1}{G_{12}} \sigma_{22}^0 T_{12} d\Omega \ \forall T_{12} \in \underset{\sim}{\Sigma} \tag{18}$$

$$\int_\Omega T_{22} \frac{\partial u_2^2}{\partial x_2} d\Omega = \int_\Omega \frac{\nu_{21}}{E_2} \sigma_{11}^0 T_{22} d\Omega \ \forall T_{22} \in \underset{\sim}{\Sigma} \tag{19}$$

$$\int_\Omega \sigma_{11}^2 \frac{\partial v_1}{\partial x_1} d\Omega + \int_\Omega \sigma_{12}^2 \frac{\partial v_2}{\partial x_2} d\Omega = 0 \tag{20}$$

$$\int_\Omega \sigma_{21}^2 \frac{\partial v_2}{\partial x_1} d\Omega + \int_\Omega \sigma_{22}^2 \frac{\partial v_2}{\partial x_2} d\Omega = 0 \tag{21}$$

We recall the most important values obtained by Talla [1]:

$$\sigma_{11}^0 = \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_{11}^0}{\partial x_1} - x_2 \frac{\partial^2 u_2^0}{\partial x_1^2} \right] \tag{22 a}$$

$$\sigma_{12}^0 = \frac{E_1}{2(1-\nu_{12}\nu_{21})} (x_2^2 - 1) \frac{\partial^3 u_2^0}{\partial x_1^3} \tag{22 b}$$

$$\sigma_{22}^0 = -\frac{E_1}{6(1-\nu_{12}\nu_{21})} (x_2^3 - 3x_2 - 2) \frac{\partial^4 u_2^0}{\partial x_1^4} + \frac{T_2^+ - T_2^-}{2} + x_2 \frac{(3-x_2^2)}{4} (T_2^+ + T_2^-) \tag{22 c}$$

to forward the calculation, lets us consider that:

$$f_1^\varepsilon = 0,\ f_2^\varepsilon = 0,\ T_1^\varepsilon = 0\ \ et\ \ T_2^\varepsilon = T_2^\varepsilon(x_1, \pm\varepsilon) \in L^2(\Gamma_\sigma^+ \cup \Gamma_\sigma^-)$$

This imply working in bending

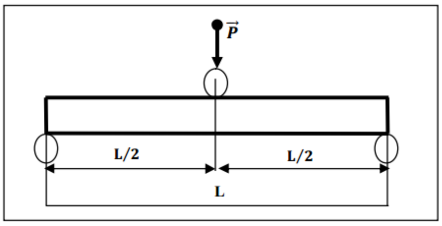

Fig. 2. Method to compensate for the indentation at supports in three-point flexure testing.

6. *Calculation de* u_1^2

According to the fundamental Lemma of mechanics, the equation (18) and (19) give respectively:

$$\begin{cases} \dfrac{\partial u_1^2}{\partial x_2} + \dfrac{\partial u_2^2}{\partial x_1} = -\dfrac{1}{G_{12}} \sigma_{12}^0 = -\dfrac{1}{G_{12}} \dfrac{E_1}{2(1-\nu_{12}\nu_{21})} (x_2^2 - 1) \dfrac{\partial^3 u_2^0}{\partial x_1^3} \\ \dfrac{\partial u_2^2}{\partial x_2} = \dfrac{\nu_{12}}{E_2} \sigma_{11}^0 = \dfrac{\nu_{12}}{E_2} \dfrac{E_1}{(1-\nu_{12}\nu_{21})} \left[\dfrac{\partial u_{11}^0}{\partial x_1} - x_2 \dfrac{\partial^2 u_2^0}{\partial x_1^2} \right] \end{cases} \tag{23}$$

The solving of the previous equation permitted us to obtain the defection in the form:

$$u_1^2 = \left[-\frac{1}{6}(\alpha + 2\beta) x_2^3 + \frac{1}{2} \alpha x_2 \right] \frac{\partial^3 u_2^0}{\partial x_1^3} + \frac{1}{2} \beta x_2^2 \frac{\partial^2 u_{11}^0}{\partial x_1^2} - x_2 \frac{\partial u_2^2}{\partial x_1} + u_{11}^2 \tag{24}$$

Where:

$$\alpha = \frac{E_1}{G_{12}(1-\nu_{12}\nu_{21})}, \text{ and } \beta = \frac{E_1}{E_2} \frac{\nu_{21}}{(1-\nu_{12}\nu_{21})}$$

This solution prompts some comments:

a). $\dfrac{\partial u_2^2}{\partial x_1} = 0$ means that the arrow u_2^2 does not depend on x_1. Clearly, all the phases admit the same deflection.

b). $u_1^2 = \left[-\dfrac{1}{6}(\alpha + 2\beta) x_2^3 + \dfrac{1}{2} \alpha x_2 \right] \dfrac{\partial^3 u_2^0}{\partial x_1^3} + \dfrac{1}{2} \beta x_2^2 \dfrac{\partial^2 u_{11}^0}{\partial x_1^2} - x_2 \dfrac{\partial u_2^2}{\partial x_1} + u_{11}^2$

u_2^2 is a constant varying according to x_2, but varies continuously from the mean line. The constant u_{11}^2 ensures continuity from one phase to another (continuity of movement u_2^2 from one phase to another).

7. *Calculation of* σ_{11}^2

From the law of behavior in mechanics, we have:

$$\begin{Bmatrix} \varepsilon_{11}^2 \\ \varepsilon_{22}^2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11}^2 \\ \sigma_{22}^2 \\ \sigma_{12}^2 \end{bmatrix} \Leftrightarrow \begin{cases} \varepsilon_{11}^2 = \dfrac{1}{E_1} \sigma_{11}^2 - \dfrac{\nu_{21}}{E_2} \sigma_{22}^2 \\ \varepsilon_{22}^2 = -\dfrac{\nu_{12}}{E_1} \sigma_{11}^2 + \dfrac{1}{E_2} \sigma_{22}^2 \\ \gamma_{12} = \dfrac{1}{G_{12}} \sigma_{12}^2 \end{cases} \tag{25}$$

It comes that:

$$\sigma_{11}^2 = \frac{E_1}{1-\nu_{12}\nu_{21}} \left[\frac{\partial u_{11}^2}{\partial x_1} - x_2 \frac{\partial^2 u_2^2}{\partial x_1^2} + \frac{1}{2} \beta x_2^2 \frac{\partial^3 u_{11}^0}{\partial x_1^3} + \left(-\frac{1}{6}(\alpha + 2\beta) x_2^3 + \frac{1}{2} \alpha x_2 \right) \frac{\partial^4 u_2^0}{\partial x_1^4} + \frac{\nu_{21}^2}{E_2} \sigma_{11}^0 \right] \tag{27}$$

8. *Calculation of* u_2^2

To evaluate u_2^2, let us addition the equation (20) and (21), it becomes that:

$$\Leftrightarrow \int_\Omega \sigma_{11}^2 \frac{\partial v_1}{\partial x_1} d\Omega + \int_\Omega \sigma_{12}^2 \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) d\Omega + \int_\Omega \sigma_{22}^2 \frac{\partial v_2}{\partial x_2} d\Omega = 0 \tag{28}$$

Let us take fields of displacement of the form:

$$(V_1, V_2) = \left(-x_2 \frac{\partial v_2}{\partial x_1}, V_2 \right),$$

Taking in account the value of the displacement below and due to the fact that the u_2^2 and V_2 doesn't depend on x_2, we have:

$$-\int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} x_2 \frac{\partial u_{11}^2}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega + \int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} x_2^2 \frac{\partial^2 u_2^2}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega - \int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{1}{2}\beta x_2^3 \frac{\partial^3 u_{11}^0}{\partial x_1^3}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega$$

$$-\int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{1}{6}(\alpha+2\beta) x_2^4 \frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega + \int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{1}{2}\alpha x_2^2 \frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega - \int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{\nu_{21}^2 E_1}{E_2(1-\nu_{12}\nu_{21})} x_2 \frac{\partial u_{11}^0}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega$$

$$\int_\Omega \frac{E_1}{1-\nu_{12}\nu_{21}} \frac{\nu_{21}^2 E_1}{E_2(1-\nu_{12}\nu_{21})} x_2^2 \frac{\partial^2 u_2^0}{\partial x_1^2}\frac{\partial^2 v_2^0}{\partial x_1^2} d\Omega = 0 \qquad (29)$$

This integral equation uniquely determines u_2^2. Let us specify that u_2^2 is the displacement in the direction and consequently the arrow taken by the structure thus requested.

9. Calculation of σ_{12}^2

From the equation:

$$\int_\Omega \sigma_{11}^2 \frac{\partial v_1}{\partial x_1} d\Omega + \int_\Omega \sigma_{12}^2 \frac{\partial v_1}{\partial x_2} d\Omega = 0,$$

and by using an integration by part, we have:

$$\sigma_{12}^2 = -\alpha_1 x_2 \frac{\partial^2 u_{11}^2}{\partial x_1^2} + \frac{1}{2}\alpha_1\left(x_2^2-1\right)\frac{\partial^3 u_2^2}{\partial x_1^3} - \alpha_1\frac{1}{6}\beta x_2^3 \frac{\partial^4 u_{11}^0}{\partial x_1^4} + \frac{1}{24}\alpha_1(\alpha+2\beta)\left(x_2^4-1\right)\frac{\partial^5 u_2^0}{\partial x_1^5}$$
$$-\frac{1}{4}\alpha_1\alpha\left(x_2^2-1\right)\frac{\partial^5 u_2^0}{\partial x_1^5} - \alpha_1^2\frac{\nu_{21}^2}{E_2} x_2 \frac{\partial^2 u_{11}^0}{\partial x_1^2} + \frac{\nu_{21}^2}{E_2}\alpha_1\sigma_{12}^0$$

10. Calculation of σ_{22}^2

Leaving from the equation (21), we obtain the following value:

$$\sigma_{22}^2 = -\int \frac{\partial \sigma_{12}^2}{\partial x_1} dx_2 + C^{ste}. \qquad (31)$$

The constant C^{ste} is chosen so as to observe the condition $\sigma_{22}^2 = -\sigma_{12}^2 = \pm T_2$ for $x_2=\pm 1$, we can, for example take:

$$C^{ste} = \frac{T_2^+ - T_2^-}{2} + \frac{x_2\left(3-x_2^2\right)}{4}\left(T_2^+ + T_2^-\right) \qquad (32)$$

Integrating equation (31) and taking into consideration equation (32), we obtain the following equation:

$$\sigma_{22}^2 = -\frac{1}{6}\alpha_1\left(x_2^3-3x_2-2\right)\frac{\partial^4 u_2^2}{\partial x_1^4} + \frac{1}{2}\alpha_1\left(x_2^2-1\right)\frac{\partial^3 u_{11}^2}{\partial x_1^3} + \frac{1}{24}\alpha_1\beta\left(x_2^4-1\right)\frac{\partial^5 u_{11}^0}{\partial x_1^5}$$
$$-\frac{1}{120}\alpha_1(\alpha+2\beta)\left(x_2^5-5x_2-4\right)\frac{\partial^6 u_2^0}{\partial x_1^6} + \frac{1}{12}\alpha_1\alpha\left(x_2^3-3x_2-2\right)\frac{\partial^6 u_2^0}{\partial x_1^6}$$
$$+\frac{1}{2}\alpha_1^2\frac{\nu_{21}^2}{E_2}\left(x_2^2-1\right)\frac{\partial^3 u_{11}^0}{\partial x_1^3} - \frac{\nu_{21}^2}{E_2}\alpha_1\left(-\frac{1}{6}\alpha_1\left(x_2^3-3x_2-2\right)\frac{\partial^4 u_2^2}{\partial x_1^4}\right) + \frac{T_2^+ - T_2^-}{2} + \frac{x_2\left(3-x_2^2\right)}{4}\left(T_2^+ + T_2^-\right) \qquad (33)$$

We can now return on the multi-layer beam into orthotropic material. Such is at least the object of the following part.

C. Application in the Bending of the Multi-Layer Orthotropic Material Beams: the Case of Sandwich Beams

We were concerned up to now by the homogeneity and isotropic of the beams. Our preoccupation in this part of work is to see up to what point one can apply the theory previously elaborated for a multi-layer structure. To this end, let us recall the principal results.

1. Definition of the problem

The multi-layer beam is submitted to a T_2 effort concentrated in its center. It rests on two supports. One proposes to calculate the displacement u_2^2, which in the concrete cases represents the arrow.

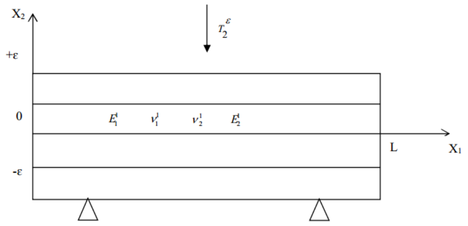

Fig. 3. Loaded beam bending at three points test [1]

This definition of the problem supposes $f_1 = f_2 = 0, T_1 = 0$ the Poisson's ratio ν_{12}, ν_{21} and the Young's modulus E_1^1, E_2^1 varies by jump.

2. Preliminaries

In order to appreciate what becomes to the equation (29), let us defined the deflection when one passes to the new configuration and let us adopt the following conventions:

- The mean line is geometrically defined by the axis ox_1;
- z_1 represent the average coast of the cell number 1;
- εh_1 is the thickness of the cell 1;
- N is the total number of cells.

$q(x_2)$ is a function depending on x_2, but constant in each cell. The relations (5.1) à (5.5) contain integrals of these types:

$$\int_{-1}^{1} q(x_2)dx_2,\ \int_{-1}^{1} x_2 q(x_2)dx_2,\ \int_{-1}^{1} x_2^2 q(x_2)dx_2,\ \int_{-1}^{1} x_2^3 q(x_2)dx_2,$$
$$\int_{-1}^{1} x_2^4 q(x_2)dx_2,$$

that we will have to evaluate. Finally, we have:

$$\sum_{l=1}^{N}\varepsilon h_1 = 2\varepsilon\ \sum_{l=1}^{N}\varepsilon h_1 = 2\varepsilon \qquad (34)$$

$$\int_{-1}^{1} q(x_2)dx_2 = \sum_{l=1}^{N} q_l h_l \qquad (35)$$

$$\int_{-1}^{1} x_2 q(x_2)dx_2 = \sum_{l=1}^{N} q_l z_1 h_1 \qquad (36)$$

$$\int_{-1}^{1} x_2^2 q(x_2)dx_2 = \frac{1}{12}\sum_{l=1}^{N} q_l\left(12z_1^2 h_1 + h_1^3\right) \qquad (37)$$

$$\int_{-1}^{1} x_2^3 q(x_2)dx_2 = \frac{1}{4}\sum_{l=1}^{N} q_1\left(4z_1^3 h_1 + z_1 h_1^3\right) \qquad (38)$$

$$\int_{-1}^{1} x_2^4 q(x_2)dx_2 = \frac{1}{80}\sum_{l=1}^{N} q_l\left(80h_1 z_1^4 + 40z_1^2 h_1^3 + h_1^5\right) \qquad (39)$$

$$\int_{-1}^{1} x_2^5 q(x_2)dx_2 = \frac{1}{16}\sum_{l=1}^{N} q_l\left(16z_1^5 h_1 + \frac{40}{3}z_1^3 h_1^3 + z_1 h_1^5\right) \qquad (40)$$

3. *Calculation of* u_2^0

Let us take again equation (29) which in this context is written as:

$$\int_\Omega \alpha_1 x_2^2 \frac{\partial^2 u_2^2}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega - \int_\Omega \alpha_1 x_2 \frac{\partial u_{11}^2}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega - \int_\Omega \alpha_1 \frac{1}{2}\beta x_2^3 \frac{\partial^3 u_{11}^0}{\partial x_1^3}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega$$
$$+\int_\Omega \left(\frac{1}{6}\alpha_1(\alpha+2\beta)x_2^4 - \frac{1}{2}\alpha_1 \alpha x_2^2\right)\frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega -$$
$$\left[\alpha_1 \frac{\nu_{21}^2}{E_2}\left\{\int_\Omega \alpha_1 x_2^2 \frac{\partial^2 u_2^0}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega - \int_\Omega \alpha_1 x_2 \frac{\partial u_{11}^0}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} d\Omega\right\}\right] = 0$$

Let us take:

$$q_1(x_2) = \frac{E_1}{1-\nu_{12}^1 \nu_{21}^1} = \alpha_1$$

and

$$d\Omega = dx_1 dx_2$$

taking into consideration the equation below, it becomes:

$$\int_0^L \frac{1}{12}\sum_{l=1}^N q_l\left(12h_1 z_1^2 + h_1^3\right)\frac{\partial^2 u_2^2}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 - \int_0^L \sum_{l=1}^N q_1 z_1 h_1 \frac{\partial u_{11}^2}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} dx_1$$
$$-\frac{1}{2}\beta\int_0^L \frac{1}{4}\sum_{l=1}^N q_1\left(4z_1^3 h_1 + z_1 h_1^3\right)\frac{\partial^3 u_{11}^0}{\partial x_1^3}\frac{\partial^2 v_2}{\partial x_1^2} dx_1$$
$$+\frac{1}{6}(\alpha+2\beta)\int_0^L \frac{1}{80}\sum_{l=1}^N q_l\left(80h_1 z_1^4 + 40z_1^2 h_1^3 + h_1^5\right)\frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1$$
$$-\frac{1}{2}\alpha\int_0^L \frac{1}{12}\sum_{l=1}^N q_l\left(12h_1 z_1^2 + h_1^3\right)\frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1$$
$$-\left[\alpha_1 \frac{\nu_{21}^2}{E_2}\left\{\int_0^L \frac{1}{12}\sum_{l=1}^N q_l\left(12h_1 z_1^2 + h_1^3\right)\frac{\partial^2 u_2^0}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 - \int_0^L \sum_{l=1}^N q_1 z_1 h_1 \frac{\partial u_{11}^0}{\partial x_1}\frac{\partial^2 v_2}{\partial x_1^2} dx_1\right\}\right] = 0 \quad (41)$$

The coupling of the effects of the membrane and of the bending effects is observed in the equation below. To uncouple the two effects, we opt for a mirror symmetry structure. Structure in which all the layers are symmetrical compared to the median plan. For such a structure, we obviously:

$$\sum_{l=1}^N q_1 z_1 h_1 = 0 \quad (42)$$

and the equation (41) becomes:

$$\int_0^L R_e \frac{\partial^2 u_2^2}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 - \int_0^L R_{e_2}\frac{\partial^3 u_{11}^0}{\partial x_1^3}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 + (\alpha+2\beta)\int_0^L R_{e_1}\frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1$$
$$-\frac{1}{2}\alpha\int_0^L R_e \frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 = \alpha_1 \frac{\nu_{21}^2}{E_2}\int_0^L R_e \frac{\partial^2 u_2^0}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 \quad (43)$$

where

$$R_e = \frac{1}{12}\sum_{l=1}^N q_l\left(12h_1 z_1^2 + h_1^3\right),$$

$$R_{e_1} = \frac{1}{480}\sum_{l=1}^N q_l\left(80h_1 z_1^4 + 40z_1^2 h_1^3 + h_1^5\right)$$

$$R_{e_2} = \frac{1}{8}\sum_{l=1}^N q_1\left(4z_1^3 h_1 + z_1 h_1^3\right),$$

are the rigidities in bending.

With the boundary conditions the problem becomes to look for u_2^2 such that:

$$\begin{cases} \int_0^L R_e \frac{\partial^2 u_2^2}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 - \int_0^L R_{e_2}\frac{\partial^3 u_{11}^0}{\partial x_1^3}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 + (\alpha+2\beta)\int_0^L R_{e_1}\frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 \\ -\frac{1}{2}\alpha\int_0^L R_e \frac{\partial^4 u_2^0}{\partial x_1^4}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 - \alpha_1 \frac{\nu_{21}^2}{E_2}\int_0^L R_e \frac{\partial^2 u_2^0}{\partial x_1^2}\frac{\partial^2 v_2}{\partial x_1^2} dx_1 = 0 \\ u_2^2 = 0 \qquad Pour\ x_1 = 0\ \ et\ \ x_1 = L \\ \frac{\partial^2 u_2^2}{\partial x_1^2} = 0 \qquad Pour\ x_1 = 0\ \ et\ \ x_1 = L \end{cases} \quad (44)$$

The second boundary condition corresponds to the beam simply posed; however other boundary conditions can be imposed. The function test $V_2 \in H_1^0(0,L)$.

A double integration by part taking into consideration the properties of the function test and the boundary conditions which leads in obtaining the variational equation in equal to:

$$\frac{\partial^4 u_2^2}{\partial x_1^4} v_2 = \alpha_1 \frac{\nu_{21}^2}{E_2}\frac{\partial^4 u_2^0}{\partial x_1^4} v_2 - \left((\alpha+2\beta)\frac{R_{e_1}}{R_e} - \frac{1}{2}\alpha\right)\frac{\partial^6 u_2^0}{\partial x_1^6} v_2, \quad (45)$$

Where $T_2 = T_2(x_1)$.

For the calculation of u_2^2, we will use the results u_2^0 already determined by Talla [1] going from the basis of a general case of the loadings applied to the upper surface of a plate. In this case, Pagano obtains an exact solution by loadings of the form:

$$T(x_1, x_2) = T_{mn}^0 \sin\left(\tfrac{m\pi x_1}{A}\right)\sin\left(\tfrac{n\pi x_2}{B}\right), \quad (46)$$

for the case of a uniform load and a concentrated loading. In this case u_2^0 take the following form:

$$u_2^0 = \sum_1^m \frac{2}{\varepsilon}\frac{L^3 T_2^\varepsilon}{\mathrm{Re}\, m^4 \pi^4}\sin\tfrac{m\pi}{L}\xi \sin\tfrac{m\pi}{L} x_1. \quad (47)$$

We replace u_2^0 with its value in the equation of u_2^2, then we obtain:

$$u_2^2 = \left[\alpha_1 \frac{\nu_{21}^2}{E_2} + A\frac{m^2\pi^2}{L^2}\right]\sum_1^m \frac{2}{\varepsilon}\frac{L^2 T_2^\varepsilon}{\mathrm{Re}\, m^4\pi^4}\sin\tfrac{m\pi}{L}\xi \sin\tfrac{m\pi}{L} x_1 \quad (48)$$

where:

$$A = \frac{2\left(\frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)} + \frac{\nu_{21}}{E_2^1}\frac{E_1^1}{\left(1-\nu_{12}^1\nu_{21}^1\right)}\right)\frac{\varepsilon^3}{960}\left[37\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1} + 2\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right] - \frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)} \times \frac{\varepsilon^3}{12}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1} + \frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}{2\frac{\varepsilon^3}{12}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1} + \frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}$$

For m fixed, the function u_2^2 varies in a linear way according to T_2^ε in a given item x_1.

It returns to the same from saying that T_2^ε is a linear function of u_2^2:

$$T_2^{\varepsilon}=\sum_1^m \frac{\varepsilon \operatorname{Re} m^4\pi^4}{\left[\alpha_1\frac{\nu_{21}^2}{E_2}+\left((\alpha+2\beta)\frac{R_{e_1}}{R_e}-\frac{1}{2}\alpha\right)\left(\frac{m\pi}{L}\right)^2\right]2L^3\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1}u_2^2, \tag{49}$$

the slope of the right-hand side being in this case is:

$$\Leftrightarrow a=\sum_1^m \frac{\varepsilon \operatorname{Re} m^4\pi^4}{\left[\alpha_1\frac{\nu_{21}^2}{E_2}+\left((\alpha+2\beta)\frac{R_{e_1}}{R_e}-\frac{1}{2}\alpha\right)\left(\frac{m\pi}{L}\right)^2\right]2L^3\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1} \tag{50}$$

4. *Calculation of* $\sigma_{11}^2, \sigma_{12}^2, \sigma_{22}^2$

The knowledge of u_2^2 enables us to evaluate the stress. In fact, from the equation (27), (30) and (33) we have respectively:

$$\sigma_{11}^2=\alpha_1\left[\frac{\partial u_{11}^2}{\partial x_1}+\frac{1}{2}\beta x_2^2\frac{\partial^3 u_{11}^0}{\partial x_1^3}+\frac{\nu_{21}^2}{E_2}\frac{\partial u_{11}^0}{\partial x_1}+\sum_L^m\left[\begin{array}{l}\alpha_1 x_2\frac{\nu_{21}^2}{E_2}(\alpha_1+1)+A\frac{m^2\pi^2}{L^2}+\\ \left(-\frac{1}{6}(\alpha+2\beta)x_2^3+\frac{1}{2}\alpha x_2\right)\end{array}\right]\frac{2}{\varepsilon}\frac{L^2T_2^{\varepsilon}}{\operatorname{Re} m^4\pi^4}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1\right] \tag{51}$$

$$\sigma_{12}^2=\alpha_1\left[-x_2\frac{\partial^2 u_{11}^2}{\partial x_1^2}-\frac{1}{6}\beta x_2^3\frac{\partial^4 u_{11}^0}{\partial x_1^4}\right]+$$
$$\alpha_1\left\{\begin{array}{l}\frac{1}{2}\alpha_1\left(1-x_2^2\right)\frac{\nu_{21}^2}{E_2}+\frac{1}{2}A\left(1-x_2^2\right)\frac{m^2\pi^2}{L^2}+\\ \frac{1}{24}\alpha_1(\alpha+2\beta)\left(x_2^4-1\right)\frac{m^2\pi^2}{L^2}-\frac{1}{4}\alpha_1\alpha\left(x_2^2-1\right)\frac{m^2\pi^2}{L^2}+\\ \alpha_1\frac{\nu_{21}^2}{E_2}\left(1-x_2^2\right)\end{array}\right\}\sum_L^m\frac{2}{\varepsilon}\frac{T_2^{\varepsilon}}{\operatorname{Re} m\pi}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1 \tag{52}$$

$$\sigma_{22}^2=\frac{1}{2}\alpha_1\left(x_2^2-1\right)\left(\frac{\partial^3 u_{11}^2}{\partial x_1^3}+\frac{1}{6}\alpha_1\frac{\nu_{21}^2}{E_2}\frac{\partial^3 u_{11}^0}{\partial x_1^3}\right)+\frac{1}{24}\alpha_1\beta\left(x_2^4-1\right)\frac{\partial^5 u_{11}^0}{\partial x_1^5}$$
$$+\frac{1}{6}\alpha_1\left(x_2^3-3x_2-2\right)\left[\begin{array}{l}\alpha_1\frac{\nu_{21}^2}{E_2}-\left(\alpha_1\frac{\nu_{21}^2}{E_2}+A\frac{m^2\pi^2}{L^2}\right)+\\ \left(\frac{1}{120}\alpha_1(\alpha+2\beta)\frac{x_2^5-5x_2-4}{x_2^3-3x_2-2}-\frac{1}{12}\alpha_1\alpha\right)\end{array}\right]\sum_L^m\frac{2}{\varepsilon}\frac{T_2^{\varepsilon}}{\operatorname{Re} L}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1+$$
$$+\frac{T_2^+-T_2^-}{2}+\frac{x_2\left(3-x_2^2\right)}{4}\left(T_2^++T_2^-\right) \tag{53}$$

Where

$$R_e=\frac{\varepsilon^3}{12}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]$$

Finally, we have the value of the deflection in the form:

$$u_2^2=\left[\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\frac{\nu_{21}^2}{E_2^1}+\left(\frac{\left(\frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)}+\frac{\nu_{21}}{E_2^1}\frac{E_1^1}{\left(1-\nu_{12}^1\nu_{21}^1\right)}\right)\frac{\varepsilon^3}{480}\left[37\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+2\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]-\frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)}\times\frac{\varepsilon^3}{12}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}{\frac{\varepsilon^3}{6}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}\right)\frac{m^2\pi^2}{L^2}\right]\sum_L^m\frac{2\times12L^3T_2^{\varepsilon}}{\varepsilon^4\pi^4m^4\left(7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1 \tag{54}$$

We base the work on the bending at the mid-span of the beam for a loading applied to the medium of the beam it is enough to make:

$$\xi=L/2 \quad et \quad x_1=L/2$$

Finally we have this:

$$u_2^2=\left[\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\frac{\nu_{21}^2}{E_2^1}+\left(\frac{\left(\frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)}+\frac{\nu_{21}}{E_2^1}\frac{E_1^1}{\left(1-\nu_{12}^1\nu_{21}^1\right)}\right)\frac{\varepsilon^3}{480}\left[37\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+2\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]-\frac{E_1^1}{G_{12}\left(1-\nu_{12}^1\nu_{21}^1\right)}\times\frac{\varepsilon^3}{12}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}{\frac{\varepsilon^3}{6}\left[7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^1}{1-\nu_{12}^2\nu_{21}^2}\right]}\right)\frac{m^2\pi^2}{L^2}\right]\sum_L^m\frac{2\times12L^3T_2^{\varepsilon}}{\varepsilon^4\pi^4m^4\left(7\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}$$

5. *Sandwich Beams at the Anhydrous State*

The sandwich beams used are made essentially of a rigid skin wood and a more flexible wood core. The skin has thickness of 2.5 mm and the core a thickness of 15 mm.

The nomenclature that we have adopted here uses three letters where the two extremes indicate the skin and the letter of the medium then indicates the core. The final value of the defection has the form:

$$u_2^2=\sum_1^m\left\{\begin{array}{l}\left[\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\frac{\nu_{21}^2}{E_2}+\left(\frac{A}{\frac{1}{6}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\right)\frac{m^2\pi^2}{L^2}\right]\times\\ \frac{24\times L^2T_2^{\varepsilon}}{\varepsilon^4m^4\pi^4\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1\end{array}\right\} \tag{56}$$

where

$$A=\left(\frac{\left(\frac{\alpha+2\beta}{768}\right)\left[62475625\frac{E_1^1}{\left(1-\nu_{12}^1\nu_{21}^1\right)}+2430000\frac{E_1^2}{\left(1-\nu_{12}^2\nu_{21}^2\right)}\right]-\alpha\frac{1}{12}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}{\frac{1}{6}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\right)$$

Finally, to return to the solution, we have to bring back our solution in the form:

$$\sigma_{ij}=\sigma_{ij}^0+\varepsilon^2\sigma_{ij}^2\Leftrightarrow\left\{\begin{array}{l}\sigma_{1j}=\sigma_{1j}^0+\varepsilon^2\sigma_{1j}^2\Leftrightarrow\left\{\begin{array}{l}\sigma_{11}=\sigma_{11}^0+\varepsilon^2\sigma_{11}^2\\ \sigma_{12}=\sigma_{12}^0+\varepsilon^2\sigma_{12}^2\end{array}\right.\\ \sigma_{2j}=\sigma_{2j}^0+\varepsilon^2\sigma_{2j}^2\Leftrightarrow\sigma_{22}=\sigma_{22}^0+\varepsilon^2\sigma_{22}^2\end{array}\right. \tag{57}$$

Expression of σ_{11}, σ_{12}, σ_{22} and U_2 becomes:

$$\sigma_{11}=\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\left[\frac{\partial u_{11}^0}{\partial x_1}+x_2\sum_1^m\frac{2T_2^{\varepsilon}L}{\operatorname{Re}\varepsilon m^2\pi^2}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1\right]+\varepsilon^2\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\left[\frac{\partial u_{11}^2}{\partial x_1}+\frac{1}{2}\beta x_2^2\frac{\partial^3 u_{11}^0}{\partial x_1^3}+\frac{\nu_{21}^2}{E_2}\frac{\partial u_{11}^0}{\partial x_1}\right]$$
$$\varepsilon^2\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\sum_L^m\left[\begin{array}{l}\alpha_1 x_2\frac{\nu_{21}^2}{E_2}(\alpha_1+1)+A\frac{m^2\pi^2}{L^2}+\\ \left(-\frac{1}{6}(\alpha+2\beta)x_2^3+\frac{1}{2}\alpha x_2\right)\end{array}\right]\frac{2}{\varepsilon}\frac{L^2T_2^{\varepsilon}}{\operatorname{Re} m^4\pi^4}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1 \tag{58}$$

$$\sigma_{12}=\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\left(1-x_2^2\right)\sum_1^m\frac{T_2^{\varepsilon}}{\operatorname{Re}\varepsilon}\sin\frac{m\pi}{L}\xi\cos\frac{m\pi}{L}x_1+\varepsilon^2\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\left[-x_2\frac{\partial^2 u_{11}^2}{\partial x_1^2}-\frac{1}{6}\beta x_2^3\frac{\partial^4 u_{11}^0}{\partial x_1^4}\right]+$$
$$\varepsilon^2\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\left\{\begin{array}{l}\frac{1}{2}\alpha_1\left(1-x_2^2\right)\frac{\nu_{21}^2}{E_2}+\frac{1}{2}A\left(1-x_2^2\right)\frac{m^2\pi^2}{L^2}+\\ \frac{1}{24}\alpha_1(\alpha+2\beta)\left(x_2^4-1\right)\frac{m^2\pi^2}{L^2}-\frac{1}{4}\alpha_1\alpha\left(x_2^2-1\right)\frac{m^2\pi^2}{L^2}+\\ \alpha_1\frac{\nu_{21}^2}{E_2}\left(1-x_2^2\right)\end{array}\right\}\sum_L^m\frac{2}{\varepsilon}\frac{T_2^{\varepsilon}}{\operatorname{Re} m\pi}\sin\frac{m\pi}{L}\xi\sin\frac{m\pi}{L}x_1 \tag{59}$$

$$\sigma_{22}^{2}=\frac{1}{2}\alpha_1\left(x_2^2-1\right)\left(\frac{\partial^3 u_{11}^2}{\partial x_1^3}+\frac{1}{6}\alpha_1\frac{\nu_{21}^2}{E_2}\frac{\partial^3 u_{11}^0}{\partial x_1^3}\right)+\frac{1}{24}\alpha_1\beta\left(x_2^4-1\right)\frac{\partial^5 u_{11}^0}{\partial x_1^5}$$
$$+\frac{1}{6}\alpha_1\left(x_2^3-3x_2-2\right)\left[\begin{array}{l}\alpha_1\frac{\nu_{21}^2}{E_2}-\left(\alpha_1\frac{\nu_{21}^2}{E_2}+A\frac{m^2\pi^2}{L^2}\right)+\\ \left(\frac{1}{120}\alpha_1\left(\alpha+2\beta\right)\frac{x_2^5-5x_2-4}{x_2^3-3x_2-2}-\frac{1}{12}\alpha_1\alpha\right)\end{array}\right]\sum_{L}^{m}\frac{2}{\varepsilon}\frac{T_2^{\varepsilon}}{\mathrm{Re}\,L}\sin\tfrac{m\pi}{L}\xi\sin\tfrac{m\pi}{L}x_1+$$
$$+\frac{T_2^{+}-T_2^{-}}{2}+\frac{x_2\left(3-x_2^2\right)}{4}\left(T_2^{+}+T_2^{-}\right)$$

(60)

$$U_2=\sum_{1}^{m}\frac{81L^3T_2^{\varepsilon}}{\varepsilon^4m^4\pi^4\left(\frac{19E_1^1}{1-\nu_{12}^1\nu_{21}^1}+8\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\sin\tfrac{m\pi}{L}\xi\sin\tfrac{m\pi}{L}x_1+$$
$$\varepsilon^2\sum_{1}^{m}\left\{\begin{array}{l}\left[\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\frac{\nu_{21}^2}{E_2}+\left(\frac{A}{\frac{1}{6}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\right)\frac{m^2\pi^2}{L^2}\right]\times\\ \frac{24\times L^2T_2^{\varepsilon}}{\varepsilon^4m^4\pi^4\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\sin\tfrac{m\pi}{L}\xi\sin\tfrac{m\pi}{L}x_1\end{array}\right\}$$

(61)

where:

$$A=\left(\frac{\left(\frac{\alpha+2\beta}{768}\right)\left[62475625\frac{E_1^1}{\left(1-\nu_{12}^1\nu_{21}^1\right)}+2430000\frac{E_1^2}{\left(1-\nu_{12}^2\nu_{21}^2\right)}\right]-\alpha\frac{1}{12}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}{\frac{1}{6}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\right)$$

$$\alpha=\frac{E_L^1}{G_{LR}^1\left(1-\nu_{LR}^1\nu_{RL}^1\right)}$$

$$\beta=\frac{E_L^1}{E_L^2}\frac{\nu_{RL}^1}{\left(1-\nu_{LR}^1\nu_{RL}^1\right)}$$

We base the work on the bending at the mid-span of the beam for a loading applied to the medium of the beam it is enough to make:

$$\xi=L/2\quad et\quad x_1=L/2$$

Finally, we have:

$$U_2=\frac{81L^3T_2^{\varepsilon}}{\varepsilon^4\pi^4\left(19\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+8\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}+$$
$$\varepsilon^2\left[\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}\frac{\nu_{21}^2}{E_2^1}+\left(\frac{A}{\frac{1}{6}\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}\right)\frac{m^2\pi^2}{L^2}\right]\times\frac{24\times L^2T_2^{\varepsilon}}{\varepsilon^4m^4\pi^4\left(4625\frac{E_1^1}{1-\nu_{12}^1\nu_{21}^1}+3375\frac{E_1^2}{1-\nu_{12}^2\nu_{21}^2}\right)}$$

(62)

III. RESULTS AND DISCUSSIONS

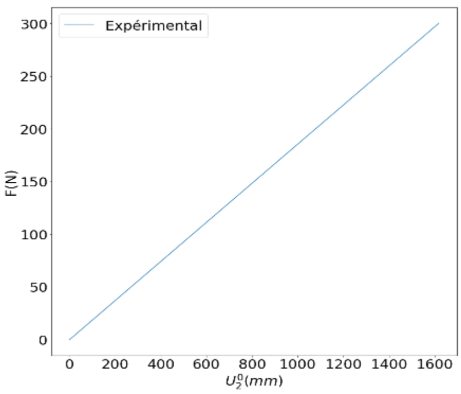

Fig. 4. Curve of first degree.

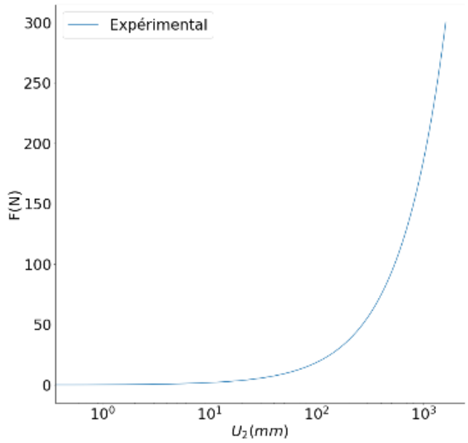

Fig. 5. Curve of second degree.

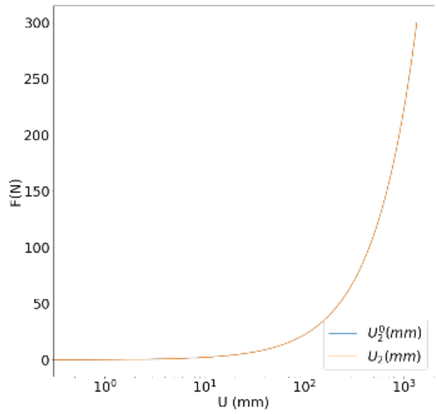

Fig. 6. Compare between the curve of the first degree and the second degree.

We can say that Talla [1] with the degree one obtained a straight line because he was using small loads represented by the figure 4. We have pushed the development to degree two and we obtain the curve in figure 5, which reflects the behaviour of the material under high stresses. We can thus say that this new approach brings a better behavior in the study of a material in the non-linear domain and disturbed by a system.

Finally, Fig. 6 shows us that on a large scale, the two curves are almost identical. This result was predictable because the asymptotic methods bring just a slight change due to the epsilon parameter.

IV. CONCLUSION

This work was devoted to the “Contribution to the development of a two-dimensional asymptotic theory of the three-point bending behaviour of multi-layered beams: applications to orthotropic phase sandwich beams”. The objective of this work was to present a methodology for analyzing the behavior in bending of the structure of sandwich beams based on the second order of asymptotic method. In the graph below, we show the representative curves of the two models for a density of ρ=0.8 g/cm^3. We have opted for the Young's modulus in the long direction; this does not detract from the generality, the other constants having the same tendency. It appears from these two curves that their slopes are negative, which proves for both models that the elastic constants decrease when humidity increases. Moreover, in the vicinity of $H\simeq 0\%$ the curve of the Foudjet-

Metangmo model is above that of the Guitard model. This difference then decreases and cancels around $H \simeq 8\%$ [8]. From this point, there is inversion and the curve of the Guitard model is above. In addition, if we set the moisture content (here we took) we see that the two approaches converge when the density decreases.

To conclude, for important loads and a better appreciation of the results, it is preferable to go to degree two to better observe the behavior of the material over time.

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